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Coherent scattering by an inhomogeneous plasma column

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Abstract. The microwave power scattered by a circular cylinder of inhomogeneous plasma contained in a glass tube is computed with the aid of the Frobenius method. It is found that computed and experimental values are in good agreement in the range $0{\cdot}2 < \lambda/R < 3{\cdot}0$. Values are also computed for wavelengths of 8 mm and 3 cm for a range of collision frequencies with the electric vector parallel or perpendicular to the cylinder axis.

1. Introduction

Coherent scattering of microwaves by a cylindrical plasma column which is homogeneous in the axial direction but has a radial density distribution has been studied many times. When the ratio of the plasma radius R to the free space wavelength λ is much greater than unity, geometrical optical theory is valid (Shmoys 1961, Heald and Wharton 1965). If R/λ is much less than unity a solution may be obtained in asymptotic form (Stratton 1941), but in the intermediate region where R/λ is near to unity the exact contour problem defined by Maxwell's equations must be solved. The Born approximation is applicable (Midzuno 1961) if the plasma permittivity is nearly equal to that of free space whereas the WKBJ (Wait 1962) method may be applied for a slowly varying electron density profile. If the plasma is enclosed in a dielectric or glass tube both approximations are difficult to apply, but solutions have been obtained (Kerker and Matijević 1961) by treating the plasma as a cylindrical sheet with quartic (Jones and Wooding 1966) or Gaussian (Lünow and Tutter 1967) electron density distribution. The method has been extended to six or eight sheets for a range of distributions of binomial form (Stanić and Wooding 1967). Another theoretical treatment made use of the self-consistent field (Faugeras 1967). In all these studies the electric vector has been parallel to the cylinder axis.

It is now shown how the problem may be solved by using the Frobenius method to obtain a solution of the wave equation with the electric vector parallel or perpendicular to the cylinder axis. It is also shown that the cylindrical sheet technique may be used in a range where the Frobenius method is invalid.

2. Theory

It is assumed that a linearly polarized plane electromagnetic wave is normally incident on a cylindrical plasma column which has an electron density distribution dependent only on the radius and which is enclosed in a glass tube. When the electromagnetic field varies in time as $\exp(-i\omega t)$, the electric and magnetic fields may be derived from the wave equations:

$$\nabla^2 \boldsymbol{E} + \omega^2 \epsilon(r)\mu_0 \boldsymbol{E} + \nabla(\boldsymbol{E} \,.\, \nabla \ln \epsilon(r)) = 0 \tag{1}$$

$$\nabla^2 \boldsymbol{H} + \omega^2 \epsilon(r)\mu_0 \boldsymbol{H} + (\nabla \ln \epsilon(r)) \times (\nabla \times \boldsymbol{H}) = 0. \tag{2}$$

For a cold plasma the spatial variation in the electron density is represented by

$$n_e(r) = n_{e0}[1-(r/R)^\gamma]$$

which leads to a relative dielectric constant

$$\epsilon_r(r) = 1 - \frac{n_e(r)}{n_c}\frac{1+i\delta}{1+\delta^2} = C_0 + C_\gamma (k_f r)^\gamma,$$

where (3)

$$C_0 = 1 - \beta_0 \frac{1+i\delta}{1+\delta^2} \qquad \text{and} \qquad C_\gamma = \beta_0 \frac{1+i\delta}{1+\delta^2}\frac{1}{(k_f R)^\gamma},$$

n_{e0} being the electron density on the cylinder axis, n_c the critical electron density at frequency ω, $\beta_0 = n_{e0}/n_c$, and $\delta = \nu/\omega$ where ν is the electron collision frequency for momentum transfer, assumed to be constant over the tube radius. We have chosen integral values of γ which permit a range of density distributions to be examined. Solutions of equations (1) and (2) are sought when the incident electromagnetic wave has its electric vector parallel or perpendicular to the cylinder axis and the electron density and collision frequency vary over a wide range. Solutions of these equations must be obtained under the conditions that the transverse components of $\boldsymbol{E}$ and $\boldsymbol{H}$ and the normal components of $\boldsymbol{D}$ and $\boldsymbol{B}$ are continuous at the plasma/glass and glass/air boundaries.

2.1. Parallel polarization

When the electric vector of the incident wave is parallel to the cylinder axis equation (1) becomes

$$\frac{1}{u}\frac{\partial}{\partial u}\left(\frac{u\partial E_z}{\partial u}\right) + \frac{1}{u^2}\frac{\partial^2 E_z}{\partial \theta^2} + \epsilon_r(u) E_z = 0$$

after introducing a new dimensionless variable $u = k_f r$, where $k_f = \omega(\epsilon_0 \mu_0)^{1/2}$, the free space wavenumber. The boundary conditions require continuity of E_z and $(\partial E_z/\partial r)/\mu$. The electric field is then given by

$$E_z = \sum_{n=-\infty}^{+\infty} R_n(u) \cos n\theta,$$

where $R_n(u)$ is a solution of the equation

$$u^2 \frac{d^2 R_n}{du^2} + u\frac{dR_n}{du} + (u^2 \epsilon_r(u) - n^2) R_n = 0 \tag{4}$$

and $n = 0, \pm 1, \pm 2, \pm 3$, etc.

Then

$$\boldsymbol{E} = (0, 0, E_z)$$

$$\boldsymbol{H} = \left(\frac{-i}{\mu_0 \omega r}\frac{\partial E_z}{\partial \theta}, \frac{i}{\mu_0 \omega}\frac{\partial E_z}{\partial r}, 0\right).$$

To an equation such as (4) the Frobenius method (Butkov 1968) may be applied with $R_n(u) = \Sigma_{j=0}^{\infty} a_j u^{j+n}$. The coefficients are determined by substituting these values of $R_n(u)$ into equation (4) with the electron density distribution given by equation (3):

$$
\begin{aligned}
a_1 &= 0 \\
a_{2k} &= -\frac{C_0 a_{2k-2} + C_\gamma a_{2k-\gamma-2}}{2k(2n+2k)} \\
a_{2k+1} &= -\frac{C_0 a_{2k-1} + C_\gamma a_{2k-\gamma-1}}{(2k+1)(2n+2k+1)} && \text{for } \gamma \text{ odd} \\
a_{2k+1} &= 0 && \text{for } \gamma \text{ even}
\end{aligned}
\tag{5}
$$

with a_0 being evaluated after imposing the boundary conditions. If instead of the electron density distribution which leads to equation (3), the Bessel type distribution

$$n_e = n_{e0} J_0(2{\cdot}405\, r/R)$$

is used, then the permittivity may be approximated by

$$\epsilon_r(r) = C_0 + C_2 u^2 + C_4 u^4 \tag{6}$$

with appropriate choice of C_0, C_2 and C_4. Then

$$
\begin{aligned}
a_1 &= 0 \\
a_{2k} &= -\frac{C_0 a_{2k-2} + C_2 a_{2k-4} + C_4 a_{2k-6}}{2k(2n+2k)} \\
a_{2k+1} &= 0.
\end{aligned}
$$

The conditions under which equation (4) leads to convergent solutions have been discussed by Piaggio (1960). The coefficients of $\partial R_n/\partial n$ and R_n must converge. Since the former coefficient is unity this presents no problem. The latter coefficient is

$$(u^2 \epsilon_r(u) - n^2)$$

which converges for all u with $\epsilon_r(u)$ specified by equation (3) for example.

2.2. Perpendicular polarization

When the incident electromagnetic wave is polarized with its electric vector perpendicular to the cylinder axis, equation (2) for the magnetic field becomes

$$\frac{1}{u}\frac{\partial}{\partial u}\left(u \frac{\partial H_z}{\partial u}\right) + \frac{1}{u^2}\frac{\partial^2 H_z}{\partial \theta^2} + \epsilon_r(u) H_z - \frac{1}{\epsilon_r(u)} + \frac{\partial \epsilon_r(u)}{\partial u}\frac{\partial H_z}{\partial u} = 0,$$

which is solved with the condition of continuity in $(\partial H_z/\partial r)/\epsilon$ and H_z at the boundaries. The magnetic field is then given by

$$H_z = \sum_{n=-\infty}^{+\infty} P_n(u) \cos n\theta,$$

where $P_n(u)$ is a solution of the equation

$$u^2 \frac{d^2 P_n}{du^2} + u\left(1 - \frac{u}{\epsilon_r(u)}\frac{d\epsilon_r(u)}{du}\right)\frac{dP_n}{du} + (u^2 \epsilon_r(u) - n^2) P_n = 0. \tag{7}$$

Then

$$H = (0, 0, H_z)$$

and

$$E = \left(\frac{\mathrm{i}}{\epsilon\omega r}\frac{\partial H_z}{\partial \theta}, \frac{-\mathrm{i}}{\epsilon\omega}\frac{\partial H_z}{\partial r}, 0\right).$$

For a collision-free plasma $\epsilon_r(r, \omega)$ may be zero at some radius r_0, but from Maxwell's equations it follows that

$$\epsilon(r, \omega)E_z = 0,$$

so in this case E_z is not necessarily zero and longitudinal electron plasma waves may be generated.

2.3. *Convergence*

If the Frobenius method is applied to equation (7) convergence is then conditional (Piaggio 1960). For a plasma which is loss-free ($\delta = \nu/\omega = 0$) and has a radial dependence given by equation (3) the solution converges for all

$$u \leqslant \left(\left|\frac{C_0}{C_\gamma}\right|\right)^{1/\gamma} \qquad C_0 \neq 0 \qquad C_\gamma \neq \infty$$

or

$$\frac{u}{k_f R} \leqslant \left(\frac{|1-\beta_0|}{\beta_0}\right)^{1/\gamma} \qquad \text{with } \beta_0 \neq 1.$$

It is limited to the range $u/(k_f R) \geqslant 1$ since a solution is required at $u = u_R = k_f R$. The range of values of u/u_R over which the solutions of equation (7) are convergent is shown in figure 1 for values of γ corresponding to various density distributions. Although solutions converge in the region below the curve the fields are required for values of $u/u_R \geqslant 1$ so that convergence occurs for $0 < \beta_0 \leqslant 0{\cdot}5$, ie for $n_{e0} \leqslant 0{\cdot}5 n_c$ which is an underdense plasma.

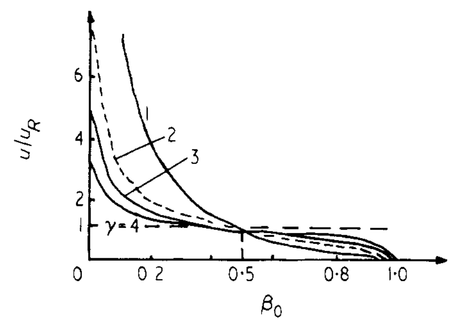

Figure 1. Curve showing convergence of solutions of equation (7) in the domain between the curve and coordinate axes for a loss-free plasma with perpendicular polarization Four values of γ are shown.

When the collision frequency is not negligible the solutions of equation (7) converge for

$$u \leqslant \left(\frac{|C_0|}{|C_\gamma|}\right)^{1/\gamma} = u_R\left(\frac{(1+\delta^2-\beta_0)^2+\delta^2\beta_0^2}{\beta_0^2(1+\delta^2)}\right)^{1/2\gamma}$$

and it follows that $u/u_R \geqslant 1$ only if

$$\beta_0 \leqslant \tfrac{1}{2}(1+\delta^2).$$

Then solutions converge in the region between the curve and the δ axis in figure 2. Hence it is seen that for a loss-free plasma the power series method cannot be applied for the case of perpendicular polarization with $\beta_0 > 0{\cdot}5$ and $\beta_0 \neq 1$, because the series is no longer convergent. However a solution may be obtained by dividing the plasma column into a number of coaxial cylindrical sheets, assuming that the electron density is constant within each sheet. In a sheet of radius R_{k-1} to R_k the permittivity $\bar{\epsilon}_r$ is given by

$$\bar{\epsilon}_r = 1 - \frac{2}{n_c(R_k^2 - R_{k-1}^2)}\int_{R_{k-1}}^{R_k} r n_e(r)\,dr.$$

This value is closer to the real situation than the arithmetic mean value that was used by Lünow and Tutter (1967) or Kerker and Matijević (1961). The boundary conditions are those used for equation (7).

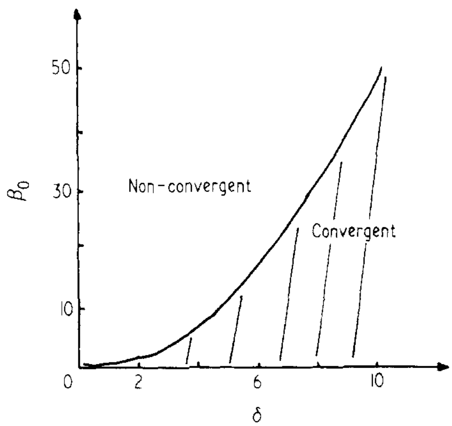

Figure 2. Values of the parameters β and δ for which the solutions of equation (7) are convergent in the region $u \leqslant u_R$ for a lossy plasma with perpendicular polarization.

2.4. *Electromagnetic fields at the receiver*

Using the radiation condition (Stratton 1941) the diffraction field at a distance from the plasma is derived as the sum of the incident and scattered fields.

2.4.1. Parallel polarization. The electric field in free space at a distance R_R from the plasma column axis is given by

$$E_z(R_R) = \sum_{n=0}^{\infty} \alpha_n(\mathrm{i}^n \mathrm{J}_n(k_f R_R) + A_n \mathrm{H}_n^{(1)}(k_f R_R)) \cos n\theta$$

where

$$\alpha_n = \begin{cases} 1 & \text{for } n = 0 \\ 2 & \text{for } n = 1, 2, \end{cases}$$

and the coefficients A_n are determined by the boundary conditions.

Then

$$A_n = \frac{\det M^{Ap}}{\det M^p},$$

where

$$M^p = \begin{vmatrix} \Omega_R^p, & -J_n(k_g R), & -Y_n(k_g R), & 0 \\ k_f \left.\dfrac{d\Omega^p}{du}\right|_{u_R}, & -k_g J_n'(k_g R), & -k_g Y_n'(k_g R), & 0 \\ 0, & J_n(k_g R_g), & Y_n(k_g R_g), & -H_n^{(1)}(k_f R_g) \\ 0, & J_n'(k_g R_g), & k_g Y_n'(k_g R_g), & -k_g H_n^{(1)\prime}(k_f R_g) \end{vmatrix}.$$

and the matrix M^{Ap} is obtained from the matrix M^p by replacement of the last two terms in the fourth column by $i^n J_n(k_f R_g)$ and $k_f i^n J_n'(k_f R_g)$. R_g and R are the external and internal radii of the glass tube, k_g is the wavenumber for glass at frequency ω, whilst J_n and Y_n are Bessel functions of order n, with $H_n^{(1)}$ a Hankel function. Differentiation with respect to the argument is represented by a prime. The function Ω_R^p is defined by

$$\Omega_R^p = \sum_{j=0}^{\infty} \frac{a_j}{a_0} (k_f R)^{n+j}$$

and the coefficient ratios a_j/a_0 are obtained from the recurrence relations (5).

2.4.2 Perpendicular polarization. The magnetic field in free space at a distance R_R from the cylinder axis is given by

$$H_z(R_R) = \sum_{n=0}^{\infty} \alpha_n (i^n J_n(k_f R_R) + B_n H_n^{(1)}(k_f R_R)) \cos n\theta,$$

with the coefficient B_n determined by the boundary conditions

$$B_n = \frac{\det M^{Bn}}{\det M^n},$$

where

$$M^n = \begin{vmatrix} \Omega_R^n, & -J_n(k_g R), & -Y_n(k_g R), & 0 \\ \dfrac{1}{k_f} \left.\dfrac{d\Omega^n}{du}\right|_{u_R}, & -\dfrac{1}{k_g} J_n'(k_g R), & -\dfrac{1}{k_g} Y_n'(k_g R), & 0 \\ 0, & J_n(k_g R_g), & Y_n(k_g R_g), & -H_n^{(1)}(k_f R_g), \\ 0, & \dfrac{1}{k_g} J_n'(k_g R_g), & \dfrac{1}{k_g} Y_n'(k_g R_g), & -\dfrac{1}{k_f} H_n^{(1)\prime}(k_f R_g) \end{vmatrix}.$$

Matrix M^{Bn} is obtained from the matrix M^n by substitution of the last two terms in the fourth column by $i^n J_n(k_f R_g)$ and $i^n J_n'(k_f R_g)/k_f$ respectively.

The function Ω_R^n represents the normalized radial part of the H_z component inside the plasma,

$$\Omega_R^n = \sum_{j=0}^{\infty} \frac{a_j}{a_0} (k_f R)^{n+j}.$$

3. Experimental and numerical results

Solutions of the scattering equation have been obtained for values of the parameter R/λ in the interval 0·1 to 10·0 and for different values of the parameter $\beta_0 = n_{e0}/n_c$, so that graphs of P_θ/P_0 are plotted as a function of electron density, where P_0 is the diffracted power at $\theta = 0°$ and P_θ is the power scattered at angle θ. The maximum values of R/λ and n_{e0}/n_c for which the problem could be solved numerically are given by

$$\left(\frac{R}{\lambda}\right)_{\max} \left(\frac{n_{\epsilon 0}}{n_c}\right)_{\max} \leqslant 2000$$

for parallel polarization.

The computed scattered power is quoted in decibels if the range of values is large, the ratio being normalized to the power diffracted into the forward direction. Experimental values of the scattered power are traced from oscillograms and plotted against time. Since the electron density falls approximately exponentially in time, the computed values are plotted against log β_0 to facilitate comparison.

Numerical values of the scattered power have been compared with experimental results obtained by scattering a collimated beam of 10 GHz or 35 GHz microwaves from an argon afterglow plasma. Computed values of the power scattered by homogeneous and inhomogeneous plasma columns are shown in figure 3. For an underdense plasma

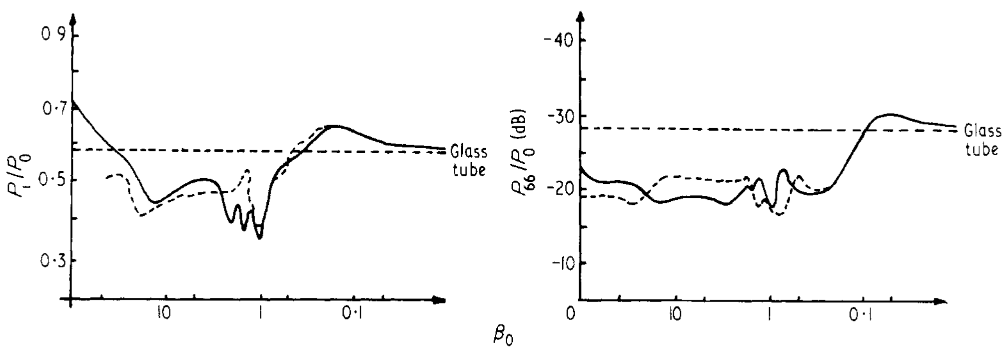

Figure 3. Influence of inhomogeneity of the loss-free plasma column on the computed scattered and diffracted powers for parallel polarization. —— homogeneous plasma; ---- inhomogeneous plasma (Bessel distribution); $R/\lambda = 1{\cdot}04$; P_i: incident electromagnetic power; P_0: diffracted EM power at $\theta = 0°$; P_{66}: scattered EM power at $\theta = 66°$.

column the profiles of scattered power are of the same shape for both cases because of the presence of the glass tube which masks the influence of inhomogeneity on the scattered profiles. For an overdense plasma the scattered profiles are quite different and the inhomogeneity is dominant. Comparing these computed results with experimental ones obtained with 35 GHz radiation it may be seen in figure 4 that the real plasma column is inhomogeneous.

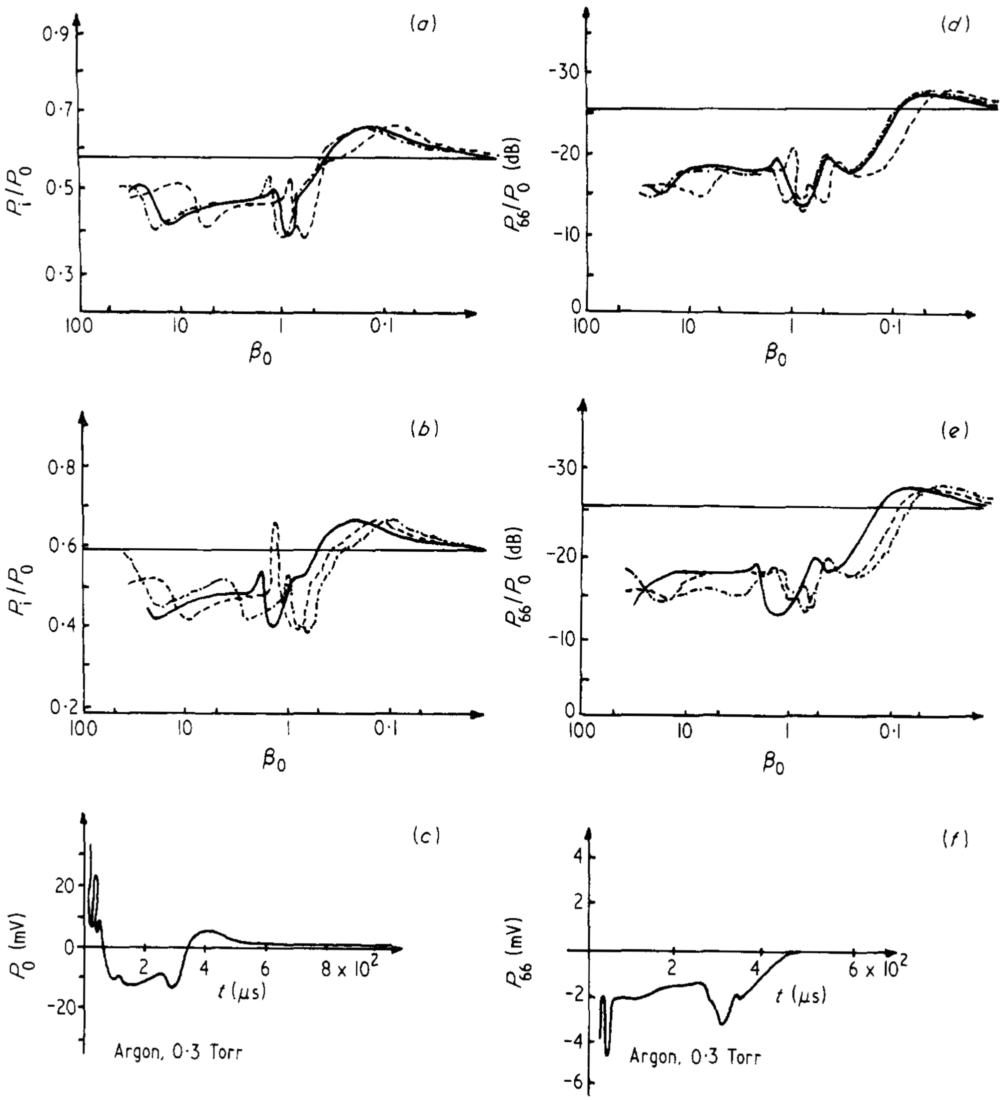

Figure 4. Comparison of experimental (c, f) values of 35 GHz radiation scattered by a plasma column with those computed (a, b, d, e) for various electron density distributions. The full horizontal line represents the level of the power scattered by the glass tube. Parallel polarization and R/λ = 1·04. Electron density distribution (a) and (d) —— cosine; – – – – quartic; – · – · – · – Bessel; (b) and (e) —— linear; – – – – parabolic $n_e = n_{e0}(1-r^2/R^2)$; – · – · – · – parabolic $n_e = n_{e0}(1{\cdot}07r^2/R^2)$; P_i is the incident power and P_θ that scattered at θ degrees.

For the case $R/\lambda = 1{\cdot}04$, scattered power profiles are computed for six possible electron density distributions in a loss-free plasma column. Comparing these computed results with a large number of experimental curves for the power scattered by the afterglow plasma it is seen in figure 4 that the closest fit is obtained with an electron density profile of the Bessel type. This is to be expected since the dominant loss process in these plasmas is ambipolar diffusion which results in this type of distribution (Brown 1959, p 49). For the same value of R/λ diffracted power at $\theta = 0°$ is computed for a lossy plasma and displayed in figure 5. Comparison of these curves with the experimental results indicates that the collision frequency is negligibly small in the afterglow argon plasma studied here.

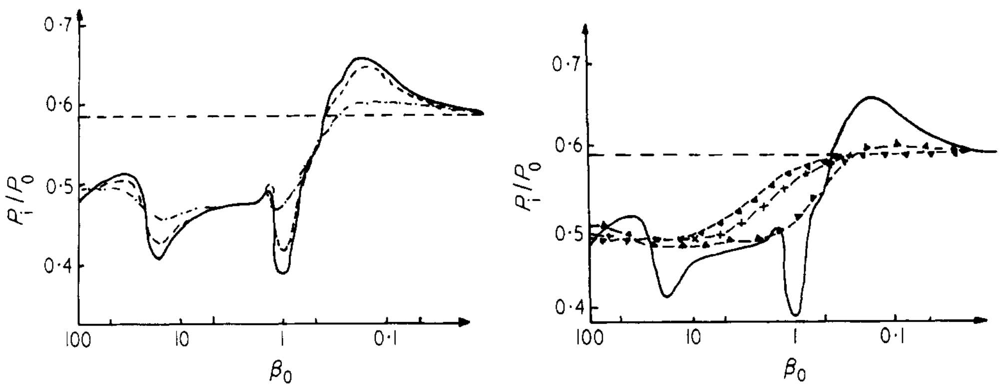

Figure 5. Influence of the collision frequency on the diffracted power at $\theta = 0°$ with parallel polarization and $R/\lambda = 1{\cdot}04$. The broken straight line at $P_i/P_0 = 0{\cdot}58$ is the power diffracted by the glass tube alone. —— $\delta = 0{\cdot}0$; ---- $\delta = 0{\cdot}1$; –·–·–·– $\delta = 0{\cdot}5$; -▲-▲-▲- $\delta = 1{\cdot}0$; -×-×-×- $\delta = 5{\cdot}0$; -▼-▼-▼- $\delta = 10{\cdot}0$.

In figures 6 and 7 are shown the numerical and experimental results for different values of the parameter R/λ. In all of these cases there is reasonable agreement between numerical and experimental results.

These numerical and experimental curves may be used in a diagnostic technique for the determination of the time variation of electron density in an afterglow plasma (Jones and Wooding 1965). For the values $R/\lambda \simeq 1$ as in figures 4, 5, 6, the rich structure of scattered power curves provides much more information about the plasma than in other cases. Computations for $R/\lambda = 2{\cdot}3$ and $R/\lambda = 2{\cdot}92$ approach the limit of validity of the geometrical optical approximation. It is seen that the power scattered by an overdense plasma is constant for some time although the electron concentration must be decaying, the reason being that the microwave beam does not penetrate completely through glass tube and plasma column, and these curves resemble the cut-off curves obtained with overdense plasma in a microwave interferometer.

When the parameter β_0 exceeds 0·5 the sheet technique is used to compute numerical values of the power scattered at various angles and for $\gamma = 1, 2$ and 4. It is seen then in figure 8 that the electron density distribution has a marked effect on the scattered power. Large peaks occur near to 110° with $\gamma = 1$ or 4 and there is some similarity in the variation with angle of the scattering below $\theta = 80°$ for $\gamma = 1$ and $\gamma = 2$. Experimental values for the scattered power have not been obtained with perpendicular polarization.

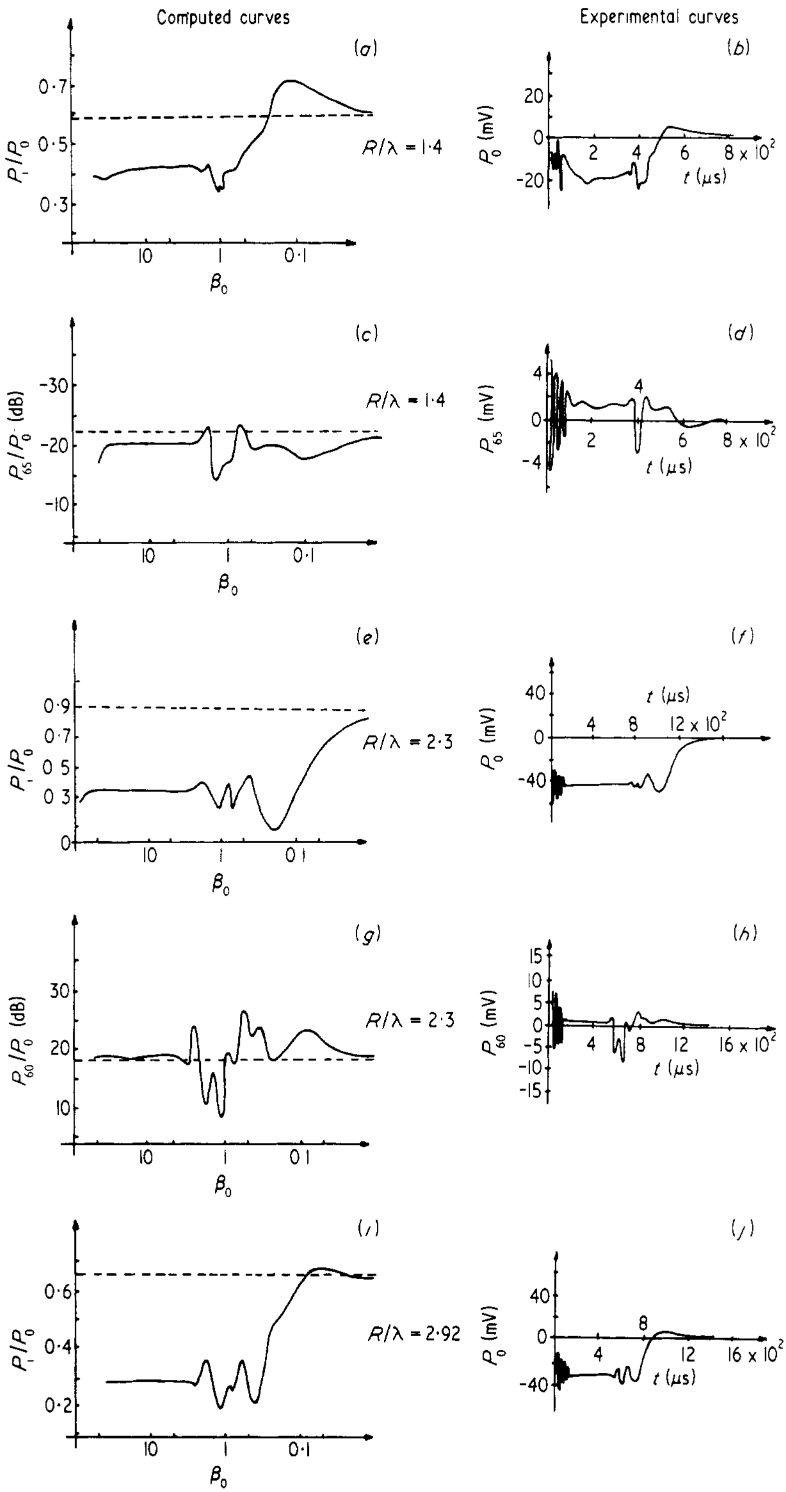

Figure 6. Comparison of computed results (a, c, e, g, i) for the Bessel type of electron density distribution with the experimental results (b, d, f, h, j) in an argon afterglow at 0·3 Torr. The incident radiation at 35 GHz is polarized parallel to the cylinder axis. The broken straight line represents the power scattered by the glass tube alone.

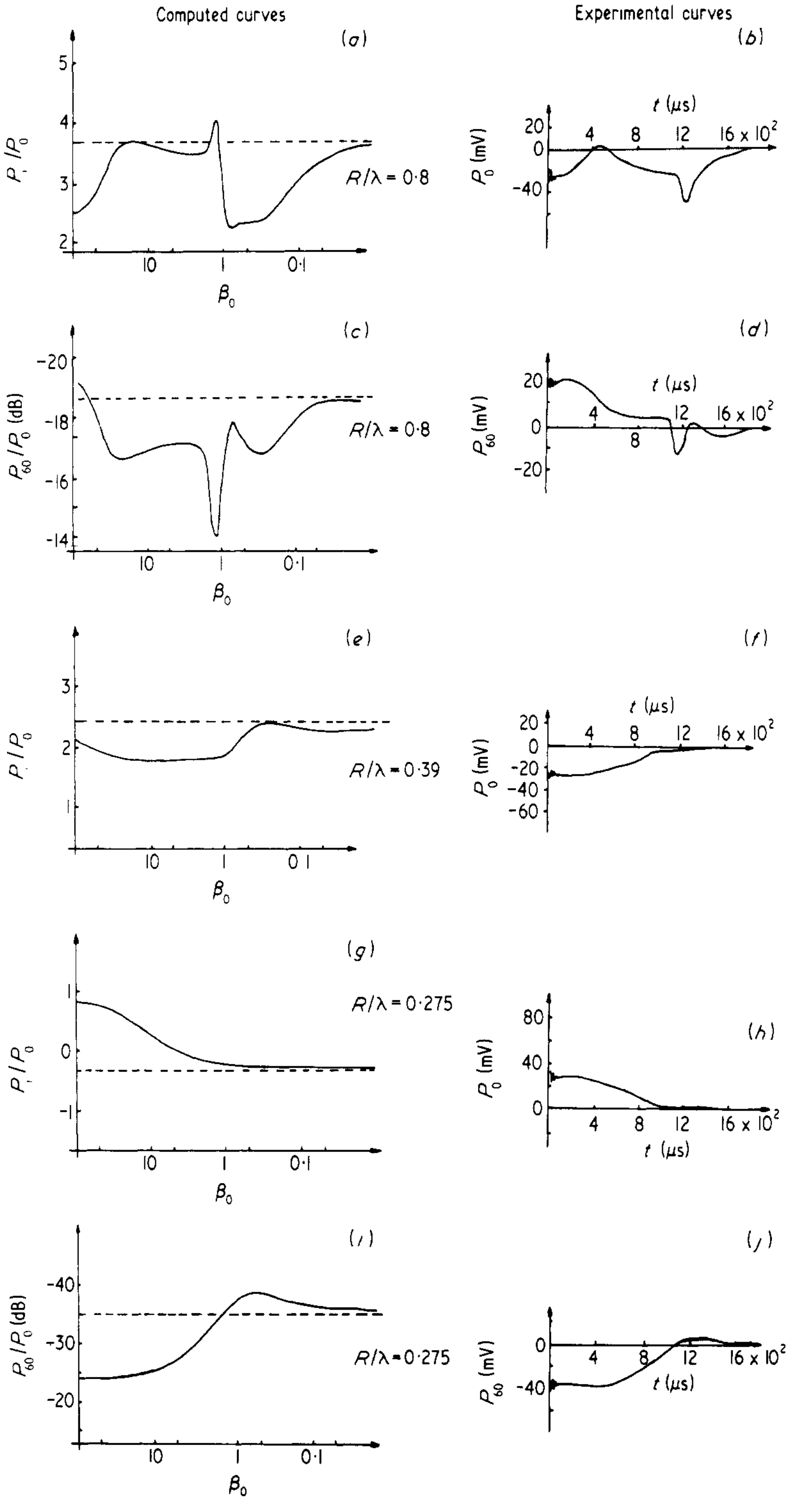

Figure 7. Comparison of computed results (*a, c, e, g, i*) for the Bessel type of electron density distribution with the experimental results (*b, d, f, h, j*) for an afterglow plasma in argon at an initial pressure of 1 Torr. The incident radiation of 10 GHz was polarized parallel to the cylinder axis.

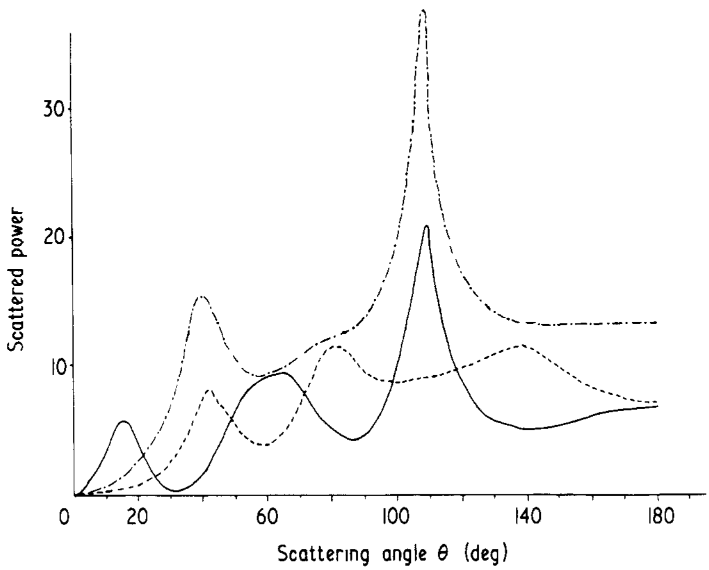

Figure 8. Angular variation of the power scattered by an inhomogeneous plasma column in a glass tube. Solution by cylindrical sheet technique. Electric field vector polarized perpendicular to cylinder axis. $\beta_0 = 0{\cdot}875$. $F = 9{\cdot}6$ GHz. $\gamma = 1$——; $\gamma = 2$- - - -; $\gamma = 4$ – · – · – · –.

4. Conclusion

Using the Frobenius method it has been shown that for the case of parallel polarization it is possible to solve the scattering equation for a wide range of values of parameters R/λ and β_0, the practical limit occurring at about $R/\lambda > 3$. For perpendicular polarization the convergence region has been found theoretically as a function of β_0. Experimental results obtained with the X and Q band microwaves show a reasonable agreement with the computed results for the case of parallel polarization. It is seen in the computed results that the scattered power distribution and variation in time are very sensitive to the value of γ.

The computations provide a useful technique for determining the electron density profile in a plasma (Jones and Wooding 1965, Jones *et al* 1968).

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